Kinematics of an elliptical orbital motion resulting from the superposition of two concentric circular movements running in opposite directions

by
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Abstract

The change in the orbital movement that occurs with water waves (from approximately circular in deep water to increasingly elliptical with decreasing water depth) deviates from the conservative view of the shoaling process such that it is accompanied *a priori* by the phenomenon of reflection. Thereby the long principal axes of the ellipse grow at the expense of the short principal axes.

For deep water conditions ($d/L \geq 0.5$), according to the theory ERR used by the author [1], for example, the circular movement on that of the diameter D_1 (= wave height H) on the water surface is superimposed on that of a smaller circular diameter D_2. The familiar relationship applies

$$D_2 = D_1 \cdot e^{-2\pi d/L}$$

with d/L as the ratio of the water depth d to the wavelength L.

The phenomenon of exponentially reduced reflection (ERR) [2] is taken into account by the fact that the water depth $d < L/2$ is assumed for the reflection and accordingly a reversal of the direction of rotation of the orbital movement occurs. At a given water depth (mirror depth) $d_2 < L/2$, the circular orbital movement of the diameter D_1 on the water surface would have to be superimposed by the orbital movement of diameter D_2, which results from the above exponential law at twice the distance $2d_2$ from the water surface.

The theory at hand is checked on the one hand using the example of the calculation of the kinematics for water waves with regard to the dimensioning of offshore structures [3] and on the other hand the example of the earth's elliptical orbit around the sun is placed alongside the results of the well-known epicyclic theory.

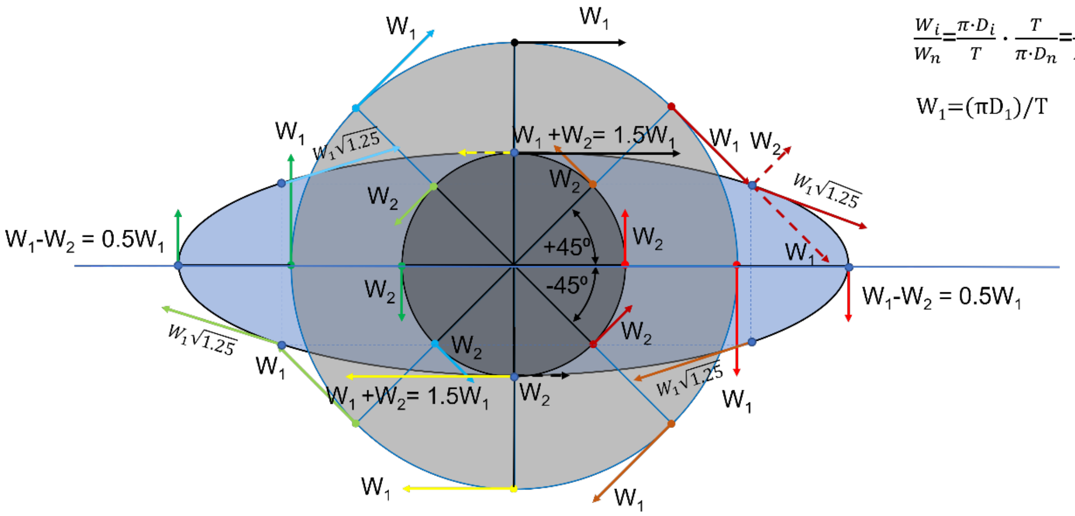
Keywords

Theory of epicycles, movement on elliptical orbits, orbital velocity, orbital acceleration, exponentially reduced reflection (ERR), orbital kinematics of water waves, elliptical earth orbit (Kepler–Ellipse).

1. Introduction

As is well known from astronomy, according to the theory of epicycles, the path of an ellipse can be composed of two circular movements, if they have opposite orbital directions and equal angular velocities ω. Thereby the smaller circle moves with its center on the circumference of the larger circle. Such described ellipses are also the basis for the results of Kepler concerning his theorems on planetary and stellar motions. The author's method for describing elliptical motion, presented here, is also based on using two circular movements with diameters $D_2 \leq D_1$, equal angular velocities ω, and opposite orbital sense.

However, these are *concentric circular movements*, of which the circumferential velocity of the smaller circle is used as a reduced mirror image of the circumferential velocity of the larger circle. In this case, there are two orbital movements with different directions of rotation, whose circumferential velocities added vectorially result in the tangent vectors of an ellipse, see Fig. 1.



$$\frac{W_i}{W_n} = \frac{\pi \cdot D_i}{T} \cdot \frac{T}{\pi \cdot D_n} = \frac{D_i}{D_n} \quad n \neq i$$

$$W_1 = (\pi D_1)/T \quad W_1 = 2W_2$$

Fig.1: Principal representation of an ellipse in normal position. Its shape results from the totality of the points of contact of its tangents. The latter are obtained from the addition of the circumferential velocity vectors of the circles with the diameters D_1 and D_2.

The shape of the searched ellipse results from the totality of contact points of its tangent vectors. The latter are obtained from the addition of the circumferential velocity vectors W_1 and W_2 belonging to each other, whose ratio of their amounts results in the example of Fig.1 from $W_1 = 2W_2$.

In this case, 8 contact points of the tangent vectors are assumed to be uniformly distributed on the respective circumference. The circumferential velocity vectors W_1 and W_2, each marked with the same colors, are determined either by exact computational vector addition or preferably sufficiently exact graphical vector addition. As an example of the vectors marked *dark red*, the latter is shown for the positions mirrored at the horizontal line with the angular deviations from the horizontal plus or minus 45°. In this case the amount of the resulting local orbital velocity of the ellipse is $W = W_1\sqrt{1.25}$.

- In the special case that the movements have the same diameters $D_1 = D_2$ and thus their circumferential velocities have the same amounts, a linearly polarized oscillation through the center of the circle results with maximum deflections $D = 2D_1 = 2D_2$ and maximum velocity amounts $W = 2W_1 = 2W_2$.

- On the other hand, in case $D_2 = 0$, the motion is on the circle with diameter D_1.

In fact, with the method presented here, the same ellipse is obtained as it is delivered by the epicycle theory. The reduction of the elliptical movement to 2 concentric circular movements with different sense of rotation offers the simpler graphical representation of the ellipse and physically more understandable derivation of the parameters of orbital velocity W and orbital acceleration a.

2. Application to the orbital motion of a water wave at the water surface over limited water depth d = 35m.

As a second example, reference is made to the orbital motion of a measuring wave for a wave-loaded pile structure located in a water depth d=35m, cf. [3].

The following input data for deep water with water depths $d \geq L/2$ were assumed to be known:

- The wave period $T = 1/f = 15.1s$ as equivalent to the circular period of the orbital motions.
- The wave height $H_{max} = 30m$ equal to the orbital circle diameter $D_1 = 30m$ at the water surface and
- the orbital circle diameter $D_2 = 5.17m$, which according to the present theory is present at 2 times the local water depth $d = 2 \cdot 35m = 70m$.

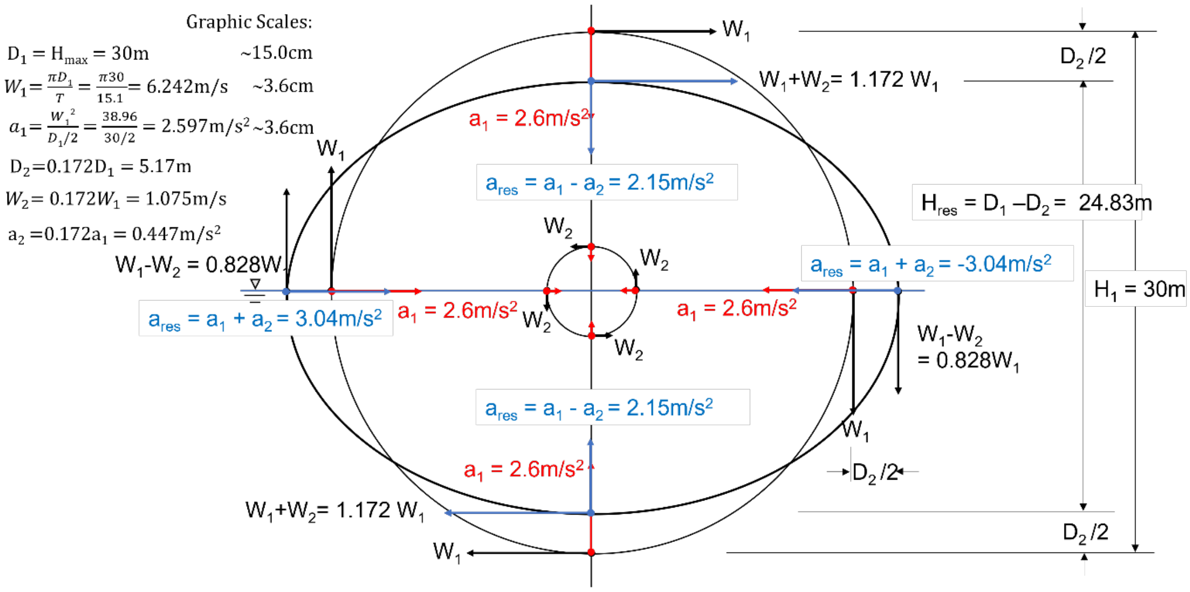


Fig.2: Graphical determination of the elliptical orbital paths in the normal position with horizontal principal axis. Vector addition for the ratio D_1/D_2 = 30m/5.17m = 5.8/1. The same ratio applies to the amounts of the circumferential velocities $W_1 = 5.8W_2$ and the circumferential accelerations $a_1 = 5.8a_2$.

In the example shown in Fig.2 it is sufficient to refer only to the 4 phase points $\pi/2$, π, $3\pi/2$ and 2π, which are decisive for the determination of the data resulting below. With the procedure not only the shape of the ellipse is determined, but for each (arbitrary) ellipse point its velocity vector and acceleration vector according to magnitude and direction also.

For the representation of the elliptical orbit only the two circle diameters are necessary:

$$D_1 = 30\text{m and } D_2 = 5.17\text{m}.$$

The lengths of the ellipse main axes result from it to:

Horizontal: $A = 2a = D_1 + D_2 = 30 + 5.17\text{m} = 35.17\text{m}$ (1)
Vertical: $B = 2b = D_1 – D_2 = 30 – 5.17\text{m} = 24.83\text{m}$ (2)

If the normal position of the ellipse is assumed, a is the horizontal long half-axis and b is the shorter half-axis perpendicular to it.

To distinguish from the designation used below for the orbital acceleration *a*, the latter is given with subscript. With the knowledge of the length of the semi-axes the ellipse can already be drawn sufficiently exactly with the help of a line program.

Analogous to the determination of the dimensions of the ellipse from the circular diameters, the orbital velocities W and orbital accelerations *a* are also obtained from the respective formulas for the two circular movements. The following magnitude ratios were confirmed based on the respective vector additions or comparison calculations.

$D_2 = 0.172D_1$ $\quad W_2 = 0.172W_1$ $\quad a_2 = 0.172a_1$ (3)

With this and the preselected orbital period T = 15.1 s, these are:

$W_1 = \frac{\pi D_1}{T} = \frac{\pi 30}{15.1} = 6.242\text{m/s}$ and $W_2 = 0.172W_1 = 1.075\text{m/s}$ (4)

$a_1 = \frac{W_1^2}{D_1/2} = \frac{38.96}{30/2} = 2.597\text{m}/s^2$ and $a_2 = 0.172a_1 = 0.447\text{m}/s^2$ (5)

The magnitudes of the resulting orbital velocity vectors W of the ellipse are obtained at the ends of the short axes as maxima from the addition of the involved circular circumferential velocities to give

horizontal maximal: $W_{res} = W_1 + W_2 = 1.172W_1 = 7.316\text{m/s}$ (6)

and at the ends of the long axes as minima from their difference vertical minimal: $W_{res} = W_1 - W_2 = 0.828W_1 = 5.168 m/s$ (7)

The maxima of the acceleration amounts a due to the determined orbital velocities result parallel to the long axes as maxima

$$a_{res} = a_1 + a_2 = \pm 3.04 m/s^2 \quad (8)$$

and as minima perpendicular

$$a_{res} = a_1 - a_2 = \pm 2.15 m/s^2. \quad (9)$$

The above results of formulas (6) to (9) agree with the results using the linear wave theory of Airy-Laplace (1842) or with the linear wave theory of "Exponentially reduced reflection" (ERR) completed by the author satisfying the theorem of conservation of mass [2].

3.Derivation of the diameters D_1 and D_2 of two concentric circular movements from the known lengths of the semi-axes of the elliptical earth orbit (Kepler–Ellipse) and their orbital period T.

In the following the present theory is put to the side of the results of the known epicycle theory at the example of the orbit of the earth around the sun. Thereby the data for the description of the earth orbit from Wikipedia, (Lemma "earth orbit"), are used.

According to the present theory the diameters D_1 and D_2 of the concentric movements are calculated from the given lengths of the ellipse main axes of the earth orbit.

The pair of equations (1) and (2) results in the pair of equations (10) and (11)

$$D_1 = B + D_2 = 2b + (a-b) = a+b \quad (10)$$

$$D_2 = (A-B)/2 = (2a - 2b)/2 = a-b \quad (11)$$

While the length a of the long earth orbital ellipse semi major axis can be taken from the Wikipedia directly

$$a = 149{,}598 \cdot 10^6 \text{ km} \quad (12)$$

with the use of the eccentricity of the earth's orbit

$$e = 0{,}0167 \cdot a \quad (13)$$

the length of the shorter semimajor axis is

$$b = 149{,}577\ 10^6 \text{ km} \quad (14)$$

Thus, the diameters of the concentric circular movements result in

$$D_1 = a+b = 299.175\ 10^6 \text{ km} \quad (15)$$

and

$$D_2 = a-b = 0.021\ 10^6 \text{ km} \quad (16)$$

Accordingly, the ratio is

$$D_1/D_2 = 299.175/0{,}021 = 14246.42 \quad (17)$$

respectively

$$D_2/D_1 = 7.019\cdot10^{-5}. \quad (18)$$

Thus, the extreme values of the orbital velocities W_{res} and the corresponding orbital accelerations a_{res} can now be determined analogously to the example shown in Fig.2.

$$D_2 = 7.019\cdot10^{-5}\cdot D_1 \qquad W_2 = 7.019\cdot10^{-5}\cdot W_1 \qquad a_2 = 7.019\cdot10^{-5}\cdot a_1 \quad (19)$$

$$D_1 = 299.175\ 10^6\ \text{km} \quad (20)$$

and

$$D_2 = 7.019\cdot10^{-5}\cdot D_1 = 0.021\cdot10^6 = 21000\ \text{km}. \quad (21)$$

According to 3rd Kepler's law, here the period of the undisturbed earth orbit is assumed to

$$T \approx 3.156\cdot10^7\ \text{s} \approx 365.2\ \text{days} \quad (22)$$

This gives the circumferential speeds as follows

$$W_1 = \frac{\pi D_1}{T} = \pi\cdot299.175\cdot10^6/31.56\cdot10^6 = 29.780925\ \text{km/s} \quad (23)$$

and

$$W_2 = 7.019\cdot10^{-5}\cdot W_1 = 7.019\cdot10^{-5}\cdot29.780925 = 0.002090\ \text{km/s} = 2{,}09\text{m/s} \quad (24)$$

The maximum orbital velocities parallel to the long major axis of the ellipse with the vertical distances ±b from the center of the ellipse are:

$$W_{res} = W_1+W_2= (1+ 7.019\cdot10^{-5})\cdot W_1 = 29.781134\ \text{km/s} \quad (25)$$

and at the ends of the long axes minimum velocities

$$W_{res} = W_1-W_2 = (1- 7.019\cdot10^{-5})\cdot W_1 = 29.778835\cdot\text{km/s} \quad (26)$$

Their arithmetic mean value = 29.7810295 km/s therefore deviates from the value, given (by Wikipedia) with 29.7859 km/s, starting from the 3rd decimal place. This may partly be due to rounding errors.

The orbital accelerations are:

$$a_1 = \frac{W_1^2}{(D_1/2)} = \frac{886.903494}{(149.587.500)} = 5.928994067\cdot\ 10^{-6}\ \text{km/s}^2 \quad (27)$$

and

$$a_2 = 7.019\cdot10^{-5}\cdot a_1 = 4.1615609\cdot\ 10^{-10}\ \text{km/s}^2. \quad (28)$$

At the moment the author cannot classify, to what extent the latter results can be useful for elliptical motions of objects in space or of elliptical motions of electrons around the atomic nucleus.

4. References

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